Dryden Research Lecture

Calculation of Turbulent Shear Flows for Atmospheric and Vortex Motions

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The basic ideas behind methods for computing the transport properties of turbulent shear flows for the past several decades are reviewed. Their shortcomings, insofar as computing a number of problems that arise in nature, are discussed. Particular emphasis is given to the case of vortex flows and atmospheric motion. The results of several years' effort towards establishing a general method for computing turbulent shear flows which does not suffer from the shortcomings of the older methods are reviewed, and a new method for the calculation of turbulent shear flows is suggested. The new results obtained when this method is applied to the computation of simple atmospheric motions and to vortex flows are presented and discussed in some detail.

Introduction

When one is asked to give a lecture such as this, the Dryden Lecture, one can proceed in one of two ways. One can either review the state of knowledge in what one supposes to be his own particular field of competence, or one can choose to present the results of one's own recent endeavors, a choice which is perhaps a little more foolhardy. Since a great portion of my professional career has been spent in wrestling with peculiarities (and I use this word advisedly) of methods for computing turbulent shear flows, I gave considerable thought to the possibility of reviewing, in as broad a way as I could, the current state-of-the-art as far as the calculation of such flows is concerned. I decided against this course of action and decided, instead, to review my own recent efforts in this direction. There were two reasons for this choice. First, there are many methods presently in use with which I am not familiar in any great detail and, second, because the art of calculating turbulent shear flows is in the process of taking a rather drastic step forward, it seemed to me an inappropriate time to review the field.*

As is the case in some other fields of technology, it is the latest generation of computers that has occasioned the present surge of new techniques in the field of shear layer computation. The power of these new machines is truly awesome. This is particularly true for one who, at the start of his professional career, spent many hours at a desk calculator solving for the point of boundary-layer separation in a diffuser using an assortment of techniques, all of which boiled down to a solution of the basic von Kármán momentum equation. The question that arose in my mind as I contemplated the use to which this new computational power should be put was, "How much more computation shall I introduce into my new formulation of the turbulent shear flow problem so that I get rid of some of the shortcomings of previous techniques but still have a program which will be useful for practical engineering problems?"

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* Since this lecture was first prepared, an excellent overview of the present situation insofar as turbulent boundary-layer computations are concerned has been prepared by W. C. Reynolds of Stanford University[1] A very complete review of the older and a few of the newer techniques for computing turbulent boundary layers is available through the published proceedings of the Stanford Conference on the Computation of Turbulent Boundary Layers.[2]

I have made my own personal choice of how far to go at this time in making use of this new computational power, and I will attempt in what follows to make this choice seem at least somewhat rational and to exhibit some of the power of the new techniques that are now within the grasp of the fluid dynamicist.

Classical Methods of Computing Turbulent Shear Flows

Before going on to describe the new technique that my colleagues and I have been developing for the calculation of strongly sheared turbulent layers, it is worthwhile pointing out what some of the shortcomings of the older methods of computing turbulent shear flows were and what might be done to eliminate these shortcomings.

In this paper I shall be dealing with essentially incompressible turbulent shear layers. The basic equation for such flows was given many years ago by Osborne Reynolds[3]; it is

$$\rho(\partial\bar{u}_i/\partial t)+\rho(\bar{u}_i\bar{u}^j)_{,j} = -\bar{p}_{,i}+(\bar{\tau}_i{}^j-\rho\langle u_i'u^{j'}\rangle)_{,j} \tag{1}$$

In this equaton, u_i is the mean velocity, u_i' the instantaneous velocity fluctuation about $\bar{u}_i$, $\bar{p}$ the mean pressure, and $\bar{\tau}_i{}^j$ the mean viscous stress given by

$$\bar{\tau}_{ij} = \mu(\bar{u}_{i,j}+\bar{u}_{j,i}) \tag{2}$$

The averaged quantity $\langle u_i'u^{j'}\rangle$ is the correlation between the i and jth component of velocity and is called the Reynolds stress tensor.

Equation (1) and the equation of continuity, namely,

$$\bar{u}_{,i}{}^i = 0 \tag{3}$$

are the equations which govern the mean motions we seek to calculate.

In solving Eqs. (1) and (3) for turbulent shear layers, it is customary to make the boundary-layer assumption. In Cartesian coordinates x, y, and z, if the components of velocity are u, v, and w, and we consider a two-dimensional shear layer with the principal velocity u and $\bar{v} = 0$, this assumption results in the following set of equations

$$\partial\bar{u}/\partial x+\partial\bar{w}/\partial z = 0 \tag{4}$$

$$\rho\frac{\partial\bar{u}}{\partial t}+\rho\bar{u}\frac{\partial\bar{u}}{\partial x}+\rho\bar{w}\frac{\partial\bar{u}}{\partial z} = -\frac{\partial\bar{p}}{\partial x}+\frac{\partial}{\partial z}\left(\mu\frac{\partial\bar{u}}{\partial z}-\rho\langle u'w'\rangle\right) \tag{5}$$

$$\partial\bar{p}/\partial z = -(\partial/\partial z)\rho\langle w'w'\rangle \tag{6}$$

We may solve for the mean velocity field $\bar{u}(x,z,t)$, $\bar{w}(x,z,t)$ if we know a set of suitable boundary conditions for Eqs. (4) and (5) and have, in addition, some knowledge of the Reynolds stress $-\rho\langle u'w'\rangle$.

Reynolds derived the equation for the general stress tensor that bears his name, namely,

$$(D/Dt)\langle u_i'u_k'\rangle = -\langle u^{j\prime}u_i'\rangle \bar{u}_{k,j} - \langle u^{j\prime}u_k'\rangle \bar{u}_{i,j} - \langle u^{j\prime}u_i'u_k'\rangle_{,j} - \rho^{-1}\langle p'u_k'\rangle_{,i} - \rho^{-1}\langle p'u_i'\rangle_{,k} + \rho^{-1}\langle p'(u'_{i,k}+u'_{k,i})\rangle + \nu g^{mn}\langle u_i'u_k'\rangle_{,mn} - 2\nu g^{mn}\langle u'_{i,m}u'_{k,n}\rangle \quad (7)$$

The boundary-layer version of this equation does not form a closed set with Eqs. (4) and (5) because of the existence of terms which contain triple velocity correlations. But even if some means of closing the system of equations through a suitable modeling of the unknown terms in Eq. (7) had been possible, the simultaneous solution of Eqs. (4), (5), and a closed version of Eq. (7) was hopelessly beyond man's capability in Reynold's day. Indeed, such an undertaking remained, for practical purposes, beyond man's capability until the last decade.

Since information about the Reynolds stress could not be obtained from Eq. (7) for use in solving Eqs. (4) and (5), the course of action was quite clear. By some means or other, it was necessary to relate the Reynolds stress $-\rho\langle u'w'\rangle$ in Eq. (5) to the profile of the mean velocity $\bar{u}$. In classical treatments of turbulent flow, it has been customary to relate the Reynolds stress to the gradient of the mean velocity profile, thus

$$\tau_t = -\rho\langle u'w'\rangle = \varepsilon\,\partial\bar{u}/\partial z \quad (8)$$

In this equation, the eddy viscosity ε has dimensions density × velocity × length. There are many formulations that have been used in specific cases of turbulent shear flow to obtain useful expressions for ε. Here we will deal only with one such formulation since this particular form will illustrate the technique and will be useful in connection with considerations we will take up presently. This formulation is the mixing length formula given by Prandtl.[4] Prandtl reasoned that if, on the average, a parcel of fluid at a given point in a turbulent shear layer had a tendency to move a distance L while retaining its original velocity, it would, on the average, produce at its new position in the shear layer a velocity fluctuation

$$u' = L\,\partial\bar{u}/\partial z + (L^2/2)\,\partial^2 u/\partial z^2 + \ldots \quad (9)$$

Since, in general, w' is of the same order of magnitude as u' in a turbulent shear flow, then one might expect the correlation $\langle u'w'\rangle$ to be proportional, to first order, to the product

$$L\,\partial\bar{u}/\partial z \cdot L\,\partial\bar{u}/\partial z$$

so that one might write

$$\tau_t = -\rho\langle u'w'\rangle = \rho l_u^2\,|\partial u/\partial z|\,\partial u/\partial z \quad (10)$$

In Eq. (10) the constant of proportionality is absorbed in the definition of the new length l_u which Prandtl called the mixing length. The absolute value sign on the first $\partial\bar{u}/\partial z$ is required so that the eddy viscosity is defined as a positive quantity, namely,

$$\varepsilon_u = \rho l_u^2\,|\partial\bar{u}/\partial z| \quad (11)$$

Equation (10) has proved to be enormously useful over the years. This would at first appear somewhat surprising since we have no more a priori information about l_u in a given situation than we do about $\langle u'w'\rangle$. The usefulness of Eq. (10) lies in the fact that many of the general features of simple turbulent shear flows can be deduced as a result of making the very simplest assumptions as to the behavior of l_u. For example, the assumption that l_u is constant across a free shear layer and proportional to the local breadth of the shear layer gives an adequate description of the nature of such layers. Again, the assumption that l_u is proportional to the distance from a solid surface permits one to deduce the logarithmic mean velocity-profile law. Thus, the real usefulness of Eq. (10) or any other mixing length formulation lies in the fact that particularly simple and intuitively pleasing forms for the behavior of l_u in a shear layer yield results that are in good agreement with experimental results.

The remarkable thing about Eq. (10) and others similar to it is that it has worked so well for most of the applications to which it has been put. This performance is remarkable because inherent in any eddy viscosity formulation is the assumption that the dynamics of the turbulent correlations $\langle u_i'u_k'\rangle$ which are given by Eq. (7) are always rapid enough so that they can "track" the development of the mean velocity profile.

The success of the eddy viscosity model is not limited to the transport of momentum alone. The idea has been successfully used for both the transport of heat and the transport of matter. For the transport of these quantities, we have the following relations:

$$\dot{q} = -\rho c_p\langle T'w'\rangle = \rho c_p l_T^2\,|\partial\bar{u}/\partial z|\,\partial\bar{T}/\partial z \quad (12)$$

and

$$\dot{m}_\alpha = -\rho\langle c_\alpha' w'\rangle = \rho l_m^2\,|\partial\bar{u}/\partial z|\,\partial\bar{c}_\alpha/\partial z \quad (13)$$

In these equations, l_T and l_m are the mixing lengths for heat and mass transport, respectively.

For flows for which gravitational effects or centrifugal forces do not play a role, l_u, l_T, and l_m are of the same order of magnitude and are generally taken to be proportional to each other. For such flows, the eddy transport model has been, in general, most successful and we must conclude that the dynamics of the correlations $\langle u'w'\rangle$, $\langle T'w'\rangle$, and $\langle c_\alpha'w'\rangle$, as defined by the appropriate

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equation for each, are indeed rapid enough to "track" the development of the mean profiles of $\bar{u}$, $\bar{T}$, and $\bar{c}_\alpha$.

There are flows in nature for which the dynamics of the second-order transport correlations are not sufficiently rapid to track the development of the mean profiles. For these flows, a simple formulation, such as the use of Eqs. (10, 12, and 13) with mixing lengths which are of the same order, is not possible. The generation of turbulence and the transport of heat and gaseous concentrations in the atmosphere are good examples of flows for which a simple mixing length formulation fails. This is especially true in the case of stable atmospheric lapse rates. The development of turbulence in the vortex trails of aircraft and the transport of heat in the turbulent vortex of an arcjet are two more examples of flows which require a more sophisticated treatment than the use of eddy transport models if the treatment is to have any generality.

Of course, the eddy transport model can be used to describe any turbulent situation if one is willing to accept Eqs. (10, 12, and 13) as definitions for l_u, l_T, and l_m.† In this case the magnitudes of l_u, l_T, and l_m will be related in a complex variety of ways to the actual scales of the various mean profiles. In spite of this, the character of any flow can be calculated by this method after one has obtained sufficient experimental data to enable one to find the required mixing lengths by empirical correlation.

Here we may ask the question, "Can we not find a somewhat more all-embracing method of turbulent transport computation so that, through the proper use of the data on turbulent mixing that are already in our possession, one can compute the general transport properties of a large variety of turbulent flows for which empirical data may not be available?"

It is towards an attempt to develop such a method that my colleagues and I have turned our attention for the past few years.

Method of Invariant Modeling

The models of turbulent shear flows that we have just discussed were based on a closure of the equations governing the flow by a modeling of the Reynolds stress term in the boundary-layer momentum equation. In order to handle cases where the dynamics of the second-order correlations are strongly dependent on the physical situation under investigation, several examples of which have just been given, we propose to close the set of equations by modeling the terms in the equation for the general stress tensor $\langle u_i'u_k'\rangle$ [Eq. (7)], which contains correlations other than the second-order correlations themselves, in terms of the quantities that are available to us in a closed model consisting of Eqs. (1) and (7). The available quantities are the mean velocity and the second-order correlations.

In obtaining expressions for the various terms in Eq. (7) which need to be modeled to effect a closure, we must be guided by some general principles. We require the modeled terms to satisfy the following criteria: 1) they must exhibit all the tensor properties and properties of symmetry of the original terms; 2) they must be dimensionally correct; 3) they must be invariant under a Galilean transformation; and 4) they must be such as to satisfy all the general conservation laws. When all these requirements are met, one finds that, if one wishes only to consider the simplest models,‡ the choice of models is not large. We will not, at this point, discuss in detail the choice of models since a discussion of this matter may be found elsewhere.[5] It will, however, be instructive to give an example of invariant modeling by deriving a model for the Reynolds stress in Eq. (1) by this method. In this case, we are seeking a closure of the equations governing the motion of a turbulent shear flow at Eq. (1). We must, therefore, seek an expression for $\langle u_i'u_k'\rangle$ in terms of the mean velocity itself. If one follows the prescription we have given above for modeling, one finds rather quickly that the tensor

$$\sigma_{ik} = \bar{u}_{i,k} + \bar{u}_{k,i} \tag{14}$$

has the required symmetry and satisfies the invariance conditions; so one might write

$$-\langle u_i'u_k'\rangle = f(\bar{u}_j)(\bar{u}_{i,k} + u_{k,i}) \tag{15}$$

In this expression, the tensor character of $\langle u_i'u_k'\rangle$ is satisfied by $\bar{u}_{i,k} + \bar{u}_{k,i}$. Thus, $f(\bar{u}_j)$ must be a scalar functional of the mean velocity and must be such that the dimensions of the resulting expression are correct. Three simple choices for $f(\bar{u}_j)$ are possible that have the required Galilean invariance. They are

$$f_1(\bar{u}_j) = \Lambda^2 \bar{u}_{,m}{}^m \tag{16}$$

$$f_2(\bar{u}_j) = \Lambda^2(\sigma_{ij}\sigma^{ij})^{1/2} \tag{17}$$

and

$$f_3(\bar{u}_j) = \Lambda^2(\Omega^m\Omega_m)^{1/2} \tag{18}$$

where

$$\Omega^m = \varepsilon^{mij}\bar{u}_{i,j} \tag{19}$$

In these expressions, Λ is a scalar measure of length and ε^{ijk} is the absolute tensor associated with the general e-symbols.[6] The quantities Ω^m defined by Eq. (19) are the contravariant components of the mean vorticity vector. The first expression given previously is zero by virtue of the continuity equation, so that we might choose to model $\langle u_i'u_k'\rangle$ as either

$$-\langle u_i'u_k'\rangle = \Lambda^2(\sigma^{mn}\sigma_{mn})^{1/2}(\bar{u}_{i,k} + \bar{u}_{k,i}) \tag{20}$$

a form which has been suggested previously by several investigators (see, for example, Lumley[7]), or as

$$-\langle u_i'u_k'\rangle = \Lambda^2(\Omega^m\Omega_m)^{1/2}(\bar{u}_{i,k} + \bar{u}_{k,i}) \tag{21}$$

If the formulas just given are reduced to the situation of the simple shear layer $\bar{u} = \bar{u}(z)$ with $\bar{v} = \bar{w} = 0$, one obtains

$$-\langle u'w'\rangle = (2)^{1/2}\Lambda^2 |\partial\bar{u}/\partial z|\, \partial\bar{u}/\partial z \tag{22}$$

and

$$-\langle u'w'\rangle = \Lambda^2 |\partial u/\partial z|\, \partial u/\partial z \tag{23}$$

Both of these expressions are similar to Prandtl's result [Eq. (10)] and may be considered generalizations of his mixing length formula. There is no way to determine which of these expressions might be more suitable to describe real motions by reference to simple shear flows. One must decide which is best by reference to another flow geometry such as a simple vortex where there is a real distinction between the scalars $\Omega^m\Omega_m$ and $\sigma^{mn}\sigma_{mn}$.

We may conclude from the expressions that we have derived above that, if we were to consider a flow for which the dynamics of the second-order correlations could "track" the mean motion, we might expect a result such as Eq. (20) or (21) to prove useful and we might expect to find that the scale parameter Λ was related to the local scale of the mean motion. To decide which formulation was the more suitable to describe physical situations, we would compare results obtained using each formulation for flows of sufficiently different geometry so that the distinction between $\Omega^m\Omega_m$ and $\sigma^{mn}\sigma_{mn}$ is exhibited.

Suppose now that we wish to apply the invariant modeling idea to the case where the dynamics of the turbulence are considered. In this case we must model the following terms in Eq. (7): $\langle u^{j\prime}u_i'u_k'\rangle$, $\langle p'(u_{i,k}' + u_{k,i}')\rangle$, $\langle p'u_k'\rangle$, and $g^{mn}\langle u_{i,m}'u_{k,n}'\rangle$. The models will be selected according to prescription that has been discussed above. There are not too many models of simple form available to us if this prescription is followed, and we make our actual choice in each case, at least for the present, on the basis of the model that appears to be the simplest general expression. The modeling that we have selected is the following:

$$\langle u_i'u_j'u_k'\rangle = -\Lambda\langle u^{m\prime}u_m'\rangle^{1/2}(\langle u_i'u_j'\rangle_{,k} + \langle u_j'u_k'\rangle_{,i} + \langle u_k'u_i'\rangle_{,j}) \tag{24}$$

$$\langle p'(u_{i,k}' + u_{k,i}')\rangle = -\rho\Lambda^{-1}\langle u^{m\prime}u_m'\rangle^{1/2}(\langle u_i'u_k'\rangle - g_{ik}\langle u^{m\prime}u_m'\rangle/3) \tag{25}$$

$$\langle p'u_k'\rangle = -\rho\Lambda\langle u^{m\prime}u_m'\rangle^{1/2}\langle u^{n\prime}u_k'\rangle_{,n} \tag{26}$$

$$g^{mn}\langle u_{i,m}'u_{k,n}'\rangle = \langle u_i'u_k'\rangle/\lambda^2 \tag{27}$$

The models just given are not new. Equations (24) and (25) are simple tensor expressions for gradient diffusion which have been used by a number of investigators to describe the diffusion of kinetic energy in a model of the equation for the contracted second-order correlation $\langle u^{m\prime}u_m'\rangle$. Equation (25) was first sug-

† Except for the case when $-\rho\langle u'v'\rangle \neq 0$ and $\partial\bar{u}/\partial y = 0$.

‡ To do otherwise would be to overtax what, at best, can be considered only an approximation to the true state of affairs.

gested by Rotta[8] in 1951. Equation (27) is a departure from the usual model of isotropic dissipation, namely,

$$g^{mn}\langle u'_{i,m}u'_{k,n}\rangle = g_{ik}\langle u^{m'}u_m'\rangle/3\lambda^2 \quad (28)$$

The choice of Eq. (27) was the result of experiencing some difficulty with the occurrence of negative autocorrelations when using Eq. (28) to try to compute the development of a turbulent boundary layer. This difficulty was eliminated by use of Eq. (27) which is a model which prohibits the production of negative autocorrelations by viscous dissipation.

It will be noted that in modeling the equation for $u_i'u_k'\rangle$ we have used two scalar lengths, Λ and λ. The single length Λ is used to model all the inviscid terms that we require. The length λ is used to model the one viscous term. More flexibility could be built into the model by choosing a different length Λ for each of the inviscid terms that are modeled. This we did not originally choose to do since it was our thought that the simpler the model, the more general would be its applicability, and it was generality that was our primary goal.¶

If the formulas just given are substituted into Eq. (7), and the equations appropriate for the decay of a freejet derived from the resulting equation considered together with Eqs. (1) and (3), it becomes clear that, if these equations are to permit self-similar freejet solutions of the type that have been found experimentally, there must be a relationship between the two length scales Λ and λ.

A simple and appropriate relationship between λ and Λ which permits the existence of similarity solutions is

$$\lambda = \Lambda/(C_1 + C_2 Re_\Lambda)^{1/2} \quad (29)$$

where

$$Re_\Lambda = \rho\langle u^{m'}u_m'\rangle^{1/2}\Lambda/\mu \quad (30)$$

We expect the inviscid scale length Λ to be related in this case to the local half-breadth δ_{fj} of the freejet. Thus, to complete our model, we would assume

$$\Lambda = C_3\delta_{fj} \quad (31)$$

In order to use the model we have just constructed, we need to determine the three basic constants, C_1, C_2, and C_3. These constants must be determined by comparing actual measurements in a freejet, for example, with computations carried out using the model. One must, therefore, by dint of much computation, conduct a three-parameter search for a set of values for C_1, C_2, and C_3 that gives the best agreement with the experimentally observed characteristics of freejets. Once such a parameter search has been completed, one should be able to use the model for predictions of the general character of any free turbulent shear flow—the decay of an isolated turbulent vortex, for example. There is one bit of ambiguity in making this transition from one modeled flow to another. This ambiguity is in regard to the definition of a local length scale. What is the length for a free vortex that is equivalent to the half-breadth of a freejet δ_{fj}? Is it the radius of the core? The answer is that one doesn't know precisely what length should be used to correspond to δ_{fj}. However, this is not so important when one is interested in the general character of the motion. In this case it is getting the order of magnitude of the length Λ correct that matters and, hence, it does not make much difference whether one chooses the core radius as a typical length for a vortex or, for that matter, twice this length. All this means is that, if we choose arbitrarily any length typical of the scale of the mean shear motion under consideration, we should expect a certain variation in the "constant" C_3 in order to get a best fit to experimental data in each case. One would not, however, expect a change in the order of magnitude of C_3 from one shear flow to the next.

The actual parameter search that was made to evaluate C_1, C_2, and C_3 for the results that will be reported here and that have been separately reported elsewhere[9,10] was not made by matching the results of computations using the model with freejet results. Such a parameter search is presently under way. The original search was made by matching the results obtained by calculation using the model with experimental results for a turbulent boundary layer on a flat plate. For this case it is necessary to modify the assumption that Λ is constant across the shear layer and equal to $C_3\delta_{bl}$. (In this case we take the boundary layer thickness δ_{bl} as a measure of the scale of the wake-like portion of the mean motion.) As explained in Ref. 9, it is necessary in order to have a meaningful boundary condition of zero turbulent fluctuation at the surface below the boundary layer to require that Λ go to zero, at least as fast as $\Lambda \sim z$. Thus, for $z \to 0$, we have assumed

$$\Lambda = C_4 z \quad (32)$$

where z is distance measured normal to the flat plate. In Ref. 6, a relationship between C_4 and C_1 was discussed. This relationship is

$$C_4 = C_1^{1/2} \quad (33)$$

so that

$$\Lambda = C_1^{1/2}z \quad (34)$$

for $z > 0$ and $\Lambda < C_3\delta_{bl}$, and

$$\Lambda = C_3\delta_{bl} \quad (35)$$

thereafter, was used as the basic model for a turbulent boundary layer.

The initial parameter search which we completed resulted in the following values for C_1, C_2, and C_3

$$C_1 = 2.5, \quad C_2 = 0.125, \quad C_3 = 0.064 \quad (36)$$

In what follows, we will use essentially these values for our model for turbulent shear flows since the results that we can discuss only briefly in this lecture are taken from more detailed studies using these numbers. These results are already available to those who wish to look further into the method proposed here.[9,10]

I do not wish to dwell here on a discussion of another method for calculating the turbulent boundary layer. There are enough "good" methods already. However, before going on to present the application of the method to two flows which are very different from ordinary boundary layers, a quick summary of how such a boundary-layer computation develops is in order.

We begin a boundary-layer computation by assuming a laminar mean-velocity boundary-layer profile $\bar{u}(z)$ at some point x_0 on a surface where the Reynolds number of the boundary layer is high enough to permit transition. At this same point, some arbitrary small disturbances are introduced into the laminar layer by assuming initial distributions of $\langle u'u'\rangle$, $\langle v'v'\rangle$, and $\langle w'w'\rangle$ as a function of z, the normal coordinate. Usually the initial shear correlation $\langle u'w'\rangle$ is assumed to be zero; i.e., $\langle u'w'\rangle \equiv 0$ at $x = x_0$. The form chosen for the distributions of $\langle u'u'\rangle$, $\langle v'v'\rangle$, $\langle w'w'\rangle$ is unimportant. The motion downstream of this initial boundary-layer profile is calculated by the simultaneous solution of a set of six partial differential equations. Without writing these out in detail, these equations are a) the continuity equation which relates the two nonzero components of mean velocity $\bar{u}$ and $\bar{w}$; b) the momentum equation which describes the behavior of $\bar{u}(z)$ as it depends on the laminar and turbulent stresses that are developed, namely, $\mu\partial\bar{u}/\partial z$ and $-\rho\langle u'w'\rangle$; c) an equation for the development of $\langle u'u'\rangle$; d) an equation for the development of $\langle v'v'\rangle$; e) an equation for the development of $\langle w'w'\rangle$; and f) an equation for the development of $\langle u'w'\rangle$.

If the Reynolds number of the initial boundary-layer profile is high enough, the disturbances $\langle u'u'\rangle$, $\langle v'v'\rangle$, and $\langle w'w'\rangle$ as well as the shear correlation $\langle u'w'\rangle$ grow rapidly as a "transition" of a sort takes place. Eventually this growth ceases and, if the proper choice of the parameters C_1, C_2, and C_3 has been made, a typical turbulent boundary-layer results.

¶ It should be pointed out that improvement and refinement of the models given and used in the computations reported here is the goal of our present research. It has already been determined, for example, that for a general model applicable to both boundary layers and free shear flows one must, for an accurate description of the profiles of both the basic mean and fluctuating quantities, consider that the length Λ in diffusive models, [Eqs. (24) and (26)], and in the model for the tendency towards isotropy [Eq. (25)], must be different.

It is not really too interesting to discuss boundary-layer calculations made in this way. The method was forced to give a pretty good boundary layer by the choice of C_1, C_2, and C_3. What is interesting is to see whether the method will give correct results and new insights into a very different type of shear flow.

In what follows we will discuss briefly the application of the method to the calculation of the decay of an isolated vortex and to the development of turbulence by shear and thermal instability in atmospheric motions. In the former case, we really have a test of the method, for the geometry of the flow is very different from that for which the parameters in the model were selected. In the latter case, we see an actual extension of the method to a more complicated flow problem and some of the insights into the nature of such flows that the method permits.

Decay of an Isolated Vortex §

If the method we have discussed in the previous sections is applied to the problem of the decay of an isolated vortex for which $\bar{u} = \bar{w} = 0$ and $\bar{v} = \bar{v}(r,t)$ where u, v, and w are the components of velocity in the radial, tangential, and axial directions, respectively, we obtain the following set of equations:

$$\partial\bar{v}/\partial t = \nu[(\partial^2\bar{v}/\partial r^2)+(1/r)\,\partial\bar{v}/\partial r-(\bar{v}/r^2)]-(\partial/\partial r)\langle u'v'\rangle-(2\langle u'v'\rangle/r) \quad (37)$$

$$\frac{\partial\langle u'u'\rangle}{\partial t} = \frac{4\bar{v}\langle u'v'\rangle}{r}+\frac{\partial M}{\partial r}\left[\frac{5\partial\langle u'u'\rangle}{\partial r}+\frac{2(\langle u'u'\rangle-\langle v'v'\rangle)}{r}\right]+ M\left[\frac{5\partial^2\langle u'u'\rangle}{\partial r^2}+\frac{5}{r}\frac{\partial\langle u'u'\rangle}{\partial r}-\frac{4}{r}\frac{\partial\langle v'v'\rangle}{\partial r}-\frac{6(\langle u'u'\rangle-\langle v'v'\rangle)}{r^2}\right]+ \frac{M}{\Lambda^2}\left(\frac{\langle v'v'\rangle+\langle w'w'\rangle-2\langle u'u'\rangle}{3}\right)-2\nu\frac{\langle u'u'\rangle}{\lambda^2}+ \nu\left[\frac{\partial^2\langle u'u'\rangle}{\partial r^2}+\frac{1}{r}\frac{\partial\langle u'u'\rangle}{\partial r}-\frac{2(\langle u'u'\rangle-\langle v'v'\rangle)}{r^2}\right] \quad (38)$$

$$\frac{\partial\langle v'v'\rangle}{\partial t} = -2\langle u'v'\rangle\left(\frac{\partial\bar{v}}{\partial r}+\frac{\bar{v}}{r}\right)+\frac{\partial M}{\partial r}\left[\frac{\partial\langle v'v'\rangle}{\partial r}+\frac{2}{r}(\langle u'u'\rangle-\langle v'v'\rangle)\right]+ M\left[\frac{\partial^2\langle v'v'\rangle}{\partial r^2}+\frac{1}{r}\frac{\partial\langle v'v'\rangle}{\partial r}+\frac{4}{r}\frac{\partial\langle u'u'\rangle}{\partial r}+\frac{6(\langle u'u'\rangle-\langle v'v'\rangle)}{r^2}\right]+ \frac{M}{\Lambda^2}\left(\frac{\langle u'u'\rangle+\langle w'w'\rangle-2\langle v'v'\rangle}{3}\right)-2\nu\frac{\langle v'v'\rangle}{\lambda^2}+ \nu\left[\frac{\partial^2\langle v'v'\rangle}{\partial r^2}+\frac{1}{r}\frac{\partial\langle v'v'\rangle}{\partial r}+\frac{2(\langle u'u'\rangle-\langle v'v'\rangle)}{r^2}\right] \quad (39)$$

$$\frac{\partial\langle w'w'\rangle}{\partial t} = \frac{\partial M}{\partial r}\frac{\partial\langle w'w'\rangle}{\partial r}+M\left(\frac{\partial^2\langle w'w'\rangle}{\partial r^2}+\frac{1}{r}\frac{\partial\langle w'w'\rangle}{\partial r}\right)+ \frac{M}{\Lambda^2}\left(\frac{\langle u'u'\rangle+\langle v'v'\rangle-2\langle w'w'\rangle}{3}\right)-2\nu\frac{\langle w'w'\rangle}{\lambda^2}+ \nu\left(\frac{\partial^2\langle w'w'\rangle}{\partial r^2}+\frac{1}{r}\frac{\partial\langle w'w'\rangle}{\partial r}\right) \quad (40)$$

$$\frac{\partial\langle u'v'\rangle}{\partial t} = -\langle u'v'\rangle\left(\frac{\partial\bar{v}}{\partial r}+\frac{\bar{v}}{r}\right)+2\langle v'v'\rangle\frac{\bar{v}}{r}+3\frac{\partial M}{\partial r}\frac{\partial\langle u'v'\rangle}{\partial r}+ M\left(3\frac{\partial^2\langle u'v'\rangle}{\partial r^2}+\frac{3}{r}\frac{\partial\langle u'v'\rangle}{\partial r}-12\frac{\langle u'v'\rangle}{r^2}\right)- \frac{M}{\Lambda^2}\langle u'v'\rangle-2\nu\frac{\langle u'v'\rangle}{\lambda^2}+\nu\left(\frac{\partial^2\langle u'v'\rangle}{\partial r^2}+\frac{1}{r}\frac{\partial\langle u'v'\rangle}{\partial r}-4\frac{\langle u'v'\rangle}{r^2}\right) \quad (41)$$

In these equations

$$M = \Lambda(\langle u'u'\rangle+\langle v'v'\rangle+\langle w'w'\rangle)^{1/2}$$

§ The results presented in this section are taken from Ref. 10 and the parameters in this case are $C_1 = 2.5$, $C_2 = 0.125$ and $C_3 = 0.16 \cdot r_1(t)$ where $r_1(t)$ is the instantaneous core size of the vortex.

The boundary conditions to apply to these equations are as follows: a) at $r \to \infty$; $r\bar{v} \to \Gamma_\infty/2\pi$

$$\langle u'u'\rangle = \langle v'v'\rangle = \langle w'w'\rangle = \langle u'v'\rangle = 0$$

b) at $r \to 0$

$$\bar{v} = \alpha r,\quad \langle u'u'\rangle = a_1+b_1r^2,\quad \langle v'v'\rangle = a_1+b_2r^2$$
$$\langle w'w'\rangle = a_3+b_3r^2,\quad \langle u'v'\rangle = \beta r^2$$

where the a's and b's and α and β are functions of time to be determined. These boundary conditions (as $r \to 0$) are consequences of the symmetry of the problem. It is not obvious that $\langle u'u'\rangle$ must equal $\langle v'v'\rangle$ at $r = 0$ from a first look at the equations for these quantities [Eqs. (38) and (39)]; nevertheless, it is a fact that the two equations become identical for $r \to 0$.

To perform a calculation of vortex decay using Eqs. (37 and 38–41), an initial distribution of turbulent energy was assumed to exist in a vortex whose initial velocity $\bar{v}(r, t_0)$ was given. The computer program is, of course, written in nondimensional form. The velocity $\bar{v}$ (along with other velocities) is made nondimensional by means of the initial core (maximum) velocity $\bar{v}_{10}$. The radius r (along with other lengths) is made nondimensional by means of the initial core radius r_{10}. The time t is made nondimensional by means of the characteristic time $r_{10}/\bar{v}_{10}$.

When this procedure is followed, the effects of viscosity in the computation are defined by the initial core Reynolds number

$$Re = \rho\bar{v}_{10}r_{10}/\mu \quad (42)$$

Its value was assumed to be 10,000.

The results may be described in terms of Figs. 1–4 where we have plotted the behavior of the nondimensional maximum velocity $\bar{v}_1/\bar{v}_{10}$, the nondimensional core radius r_1/r_{10}, the non-

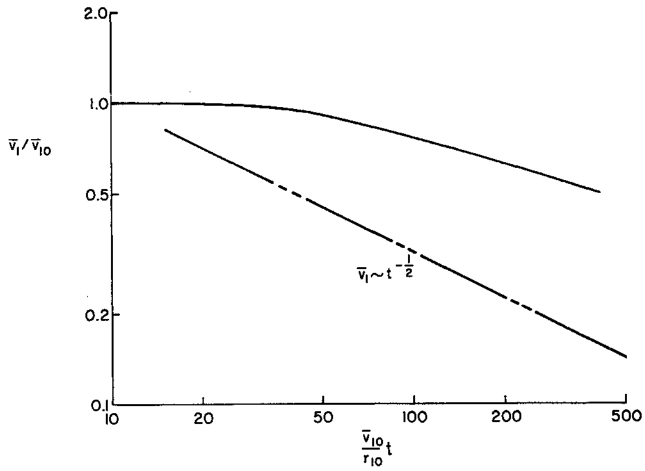

Fig. 1 Decay of the maximum or core velocity with time.

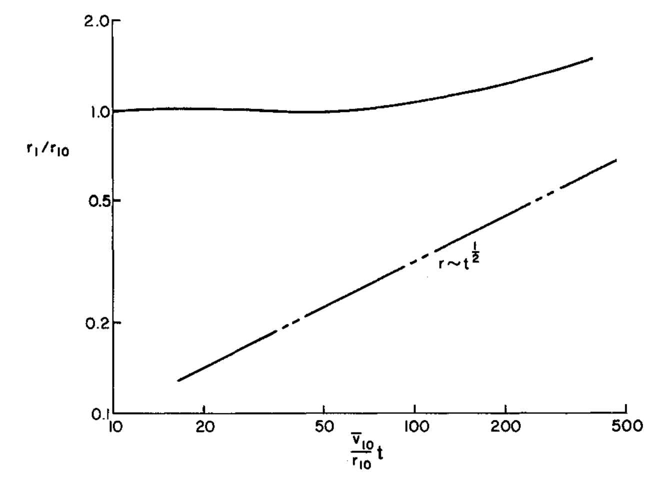

Fig. 2 Behavior of the core radius with time.

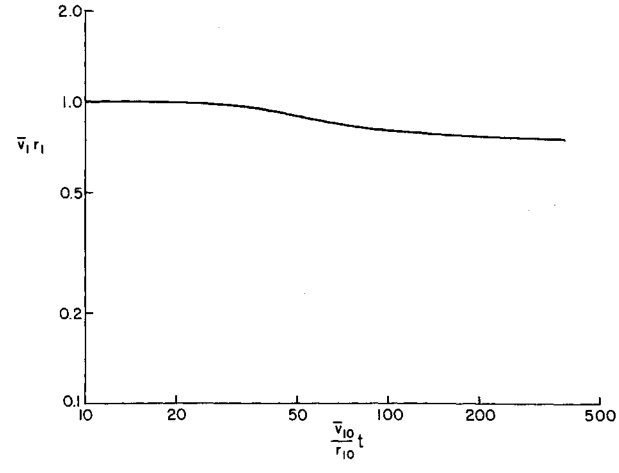

Fig. 3 Behavior of the circulation at the core radius with time.

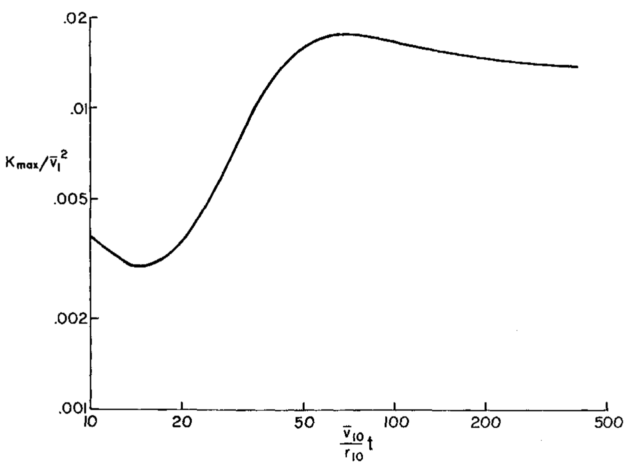

Fig. 4 Behavior of the turbulent energy parameter $K_{max}/\bar{v}_1{}^2$.

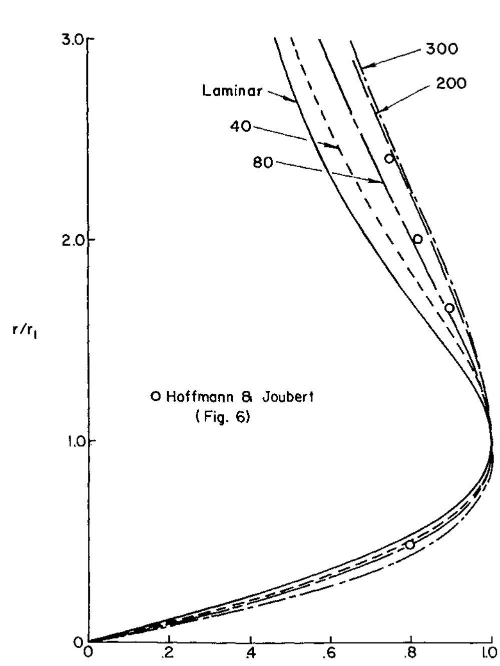

Fig. 5 Comparison of mean velocity profiles normalized to the point of maximum velocity for values of $(\bar{v}_{10}/r_{10})t$.

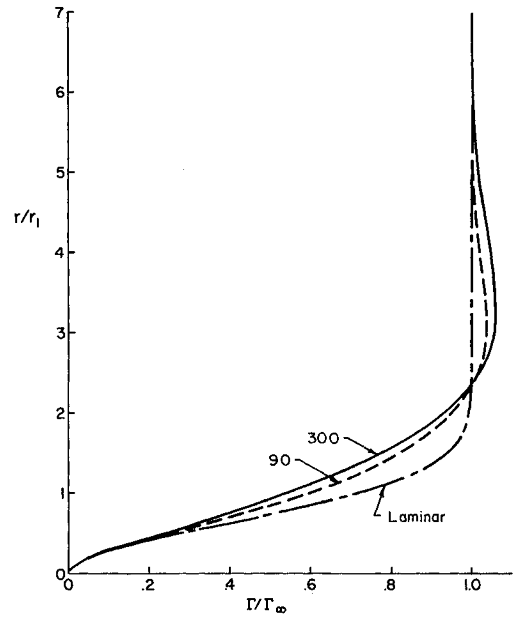

Fig. 6 Comparison of circulation profiles normalized to Γ_∞ for several values of $(\bar{v}_{10}/r_{10})t$.

dimensional circulation ratio $r_1\,\bar{v}_1/r_{10}\,\bar{v}_{10}$, and the maximum of the sum of the correlations, $K = \langle u'u'\rangle + \langle v'v'\rangle + \langle w'w'\rangle$, divided by the instantaneous value of the maximum velocity squared $\bar{v}_1{}^2$.

These results show that after an initial time of the order of fifty to one hundred times $r_{10}/\bar{v}_{10}$ the vortex core starts to grow and the maximum velocity begins to decrease in a manner suggesting that a similarity solution is being approached. If a similarity solution were to be achieved, the product $v_1\,r_1$ should approach a constant and the ratio of K_{max} to $\bar{v}_1{}^2$ should also approach a constant value. It is seen in Figs. 3 and 4 that this behavior is emerging.

We may also test for the development of a similarity solution by comparing the nondimensional velocity, circulation, turbulent stress, and turbulent energy profiles as time increases. Such comparisons are shown in Figs. 5–8. From these figures it appears that the profiles are not changing significantly with time at the

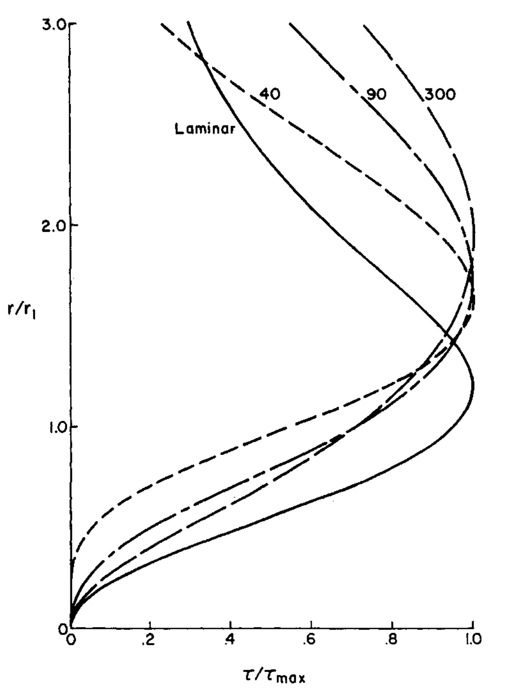

Fig. 7 Comparison of nondimensional turbulent stress profiles normalized to their maxima for several values of $(\bar{v}_{10}/r_{10})t$.

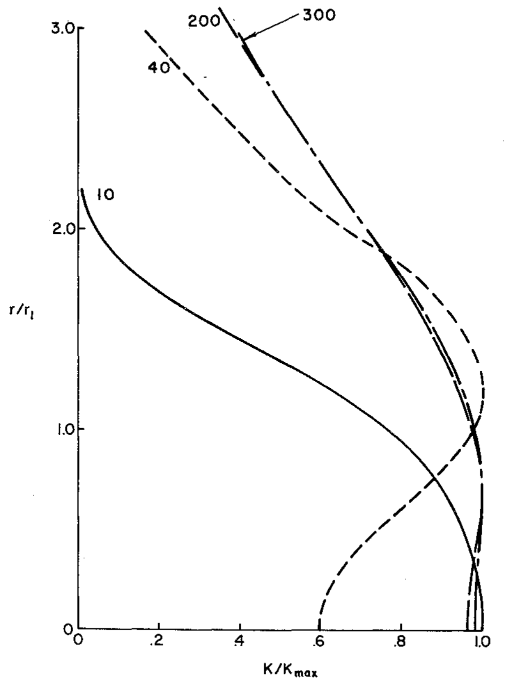

Fig. 8 Comparison of turbulent energy profiles normalized to their maxima for several values of $(\bar{v}_{10}/r_{10})t$.

later times. This is indicative of an approach to the similarity solution.

Of great interest is a comparison of the results that we have obtained with experimental results. In Fig. 5 we show a comparison of our computed mean velocity profile with some measurements of Hoffmann and Joubert.[11] It is seen that the agreement is quite good.

Two other results of our computations are in agreement with their results. At the largest nondimensional distances downstream of vortex formation observed by Hoffmann and Joubert, they observed that the ratio of the circulation at the core radius to that at infinity was 0.53 and that the ratio of the core radius r_1 to the radius where the circulation first became equal to the far field circulation r_0 was 0.34. These numbers are in general agreement with those we have obtained, namely $\Gamma_1/\Gamma_\infty = 0.55$ and $r_1/r_0 = 0.42$. It would appear that the vortex we have computed is closely related to the equilibrium vortex state being approached in Hoffmann and Joubert's experiments.

There is an interesting point to be made in conjunction with Fig. 6. We note from this figure that the circulation increases from $r/r_1 = 0$ to $r/r_1 \simeq 3$. From this point out, the circulation decreases such that this portion of the vortex is unstable in the Rayleigh sense. In our equations, when $\partial(r\bar{v})/\partial r < 0$, the production terms alone exhibit an exponential instability and they exhibit an oscillatory solution for $\partial(r\bar{v})/\partial r > 0$. This instability is limited by dissipation and diffusion at the small scale lengths we consider in these computations. The flow is, however, unstable to long wavelengths and it is well known from the experiments of Taylor[12] that the result of this instability is a breakdown to a secondary flow which forms doughnut-like ring vortices about the axis of symmetry. It is, perhaps, the instability that is inherent in this overshoot in the circulation well outside the core of a turbulent trailing vortex that is responsible for the doughnut-shaped smoke rings sometimes observed about the trailing vortices in the wakes of jet aircraft. The existence of this instability is the result of small-scale turbulent transport. The excitation of the instability by long wavelength disturbances in the atmosphere or in the trailing vortex system itself is then responsible for the production of the individual smoke rings that are observed.

The encouraging results we have obtained in applying the method of invariant modeling to the decay of an isolated vortex lead the author to believe that the method may have real usefulness in many cases where detailed experimental information concerning the turbulent flow is lacking and for which we wish to calculate the general features of the motion.

The generation of turbulence by wind shear in the atmosphere is a case in point. We will now turn to the presentation of some results obtained when the method we have been discussing is applied to such a problem.

Generation of Turbulence and Turbulent Heat Transfer in Atmospheric Shear Layers

To adapt the method of invariant modeling, as we have discussed it, to the case of a fluid in the presence of a gravitational field requires, because of a rather fortuitous quirk of nature, but little effort. It is beyond the time available in this lecture to discuss in detail the equations which govern the turbulent motion of the atmosphere. These equations are given in many standard text[13] and the modeling of those equations necessary to effect closure according to the scheme we are discussing here is given in Ref. 9 from which the results that will be considered here were taken.

The essential point is that in spite of the existence of a divergence of the turbulent velocity field, i.e., $u_{,m}^{m'} \neq 0$, no new parameters must be introduced in order to model such flows. Thus, a simple free shear layer in the atmosphere may be treated with a model that depends on only the three parameters C_1, C_2, and C_3 derived from experimental studies where a great deal of information is available.

The simple shear layer considered by the author and his colleagues in Ref. 9 is the simplest model which we believe can give meaningful results concerning the development of turbulence in the atmosphere. We assume an atmosphere in which the motion (u, v, w) in Cartesian coordinates (x, y, z) is such that $u = \bar{u}(z, t)$ and $\bar{v} = \bar{w} = 0$. Thus we assume that the mean motion consists of only one component of velocity. That component is parallel to the ground and is only a function of time and altitude. The thermodynamic state is represented by $\rho_0(z)$, the local density of the equilibrium atmosphere, and by $\bar{T}(z)$, the difference between the actual mean temperature and the value it would have in an equilibrium atmosphere. T' represents the fluctuation of temperature. In such a motion, symmetry requires that the correlations $\langle v'T'\rangle$, $\langle u'v'\rangle$, and $\langle v'w'\rangle$ be zero. To gain this amount of simplicity in the motion, it is necessary to invent a body force $X(z, t)$ which starts the motion. Actually, there is no such real force acting on the atmosphere but rather the atmosphere is driven by pressure and Coriolis forces. These forces, however, produce a far more complicated motion in the large than the one we have chosen. At the very least, one must, in such cases, consider a motion of the form $\bar{u}(z, t)$ and $\bar{w}(z, t)$. Nevertheless, these real motions can resemble locally the simple motions considered here.

When the appropriate set of modeled equations is derived for this problem, we arrive at a set of nine coupled partial differential equations of which the first two, which describe the mean state of the atmosphere, are

$$\rho_0\, \partial\bar{u}/\partial t = (\mu\, \partial^2\bar{u}/\partial z^2) - [\partial/\partial z(\rho_0\langle u'w'\rangle)] + X(z, t) \quad (43)$$

$$\rho_0\, \partial\bar{T}/\partial t = \mu\, \partial^2\bar{T}/\partial z^2 - (\partial/\partial z)(\rho_0\langle T'w'\rangle) \quad (44)$$

In addition to these equations, we need equations for the following seven quantities: $\langle u'u'\rangle$, $\langle v'v'\rangle$, $\langle w'w'\rangle$, $\langle u'w'\rangle$, $\langle u'T'\rangle$, $\langle w'T'\rangle$, and $\langle T'^2\rangle$. The equations for these correlations are

$$\frac{\partial\langle u'u'\rangle}{\partial t} = -2\langle u'w'\rangle\frac{\partial u}{\partial z} + \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 M\frac{\partial\langle u'u'\rangle}{\partial z}\right) - \frac{M}{\Lambda^2}\left(\langle u'u'\rangle - \frac{K}{3}\right) + \frac{\mu}{\rho_0}\frac{\partial^2}{\partial z^2}\langle u'u'\rangle - \frac{2\mu}{\rho_0}\frac{\langle u'u'\rangle}{\lambda^2} \quad (45)$$

$$\frac{\partial\langle v'v'\rangle}{\partial t} = \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 M\frac{\partial\langle v'v'\rangle}{\partial z}\right) - \frac{M}{\Lambda^2}\left(\langle v'v'\rangle - \frac{K}{3}\right) + \frac{\mu}{\rho_0}\frac{\partial^2}{\partial z^2}\langle v'v'\rangle - \frac{2\mu}{\rho_0}\frac{\langle v'v'\rangle}{\lambda^2} \quad (46)$$

$$\frac{\partial\langle w'w'\rangle}{\partial t}=\frac{5}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 M\frac{\partial\langle w'w'\rangle}{\partial z}\right)-\frac{M}{\Lambda^2}\left(\langle w'w'\rangle-\frac{K}{3}\right)+\frac{2}{\rho_0}M\frac{\partial\langle w'w'\rangle}{\partial z}\frac{\partial\rho_0}{\partial z}+\frac{\mu}{\rho_0}\frac{\partial^2}{\partial z^2}\langle w'w'\rangle-\frac{2}{\rho_0}\frac{\langle w'w'\rangle}{\lambda^2}-\frac{2(\mu+\mu^*)}{\rho_0}\left\{\langle w'w'\rangle\left[\frac{\partial}{\partial z}\left(\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial z}\right)-\frac{1}{\rho_0^{\,2}}\left(\frac{\partial\rho_0}{\partial z}\right)^2\right]\right\}+\frac{2g}{T_0}\langle w'T'\rangle \tag{47}$$

$$\frac{\partial\langle u'w'\rangle}{\partial t}=-\langle w'w'\rangle\frac{\partial\bar{u}}{\partial z}+\frac{3}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 M\frac{\partial}{\partial z}\langle u'w'\rangle\right)-\frac{M}{\Lambda^2}\langle u'w'\rangle+\frac{M}{\rho_0}\frac{\partial}{\partial z}\langle u'w'\rangle\frac{\partial\rho_0}{\partial z}+\frac{\mu}{\rho_0}\frac{\partial^2}{\partial z^2}\langle u'w'\rangle-\frac{2\mu}{\rho_0}\frac{\langle u'w'\rangle}{\lambda^2}-\frac{(\mu+\mu^*)}{\rho_0}\left\{\langle u'w'\rangle\frac{\partial}{\partial z}\left(\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial z}\right)-\frac{\langle u'w'\rangle}{\rho_0^{\,2}}\left(\frac{\partial\rho_0}{\partial z}\right)^2\right\}+\frac{g}{T_0}\langle u'T'\rangle \tag{48}$$

$$\frac{\partial\langle u'T'\rangle}{\partial t}=-\langle u'w'\rangle\frac{\partial\bar{T}}{\partial z}-\langle w'T'\rangle\frac{\partial\bar{u}}{\partial z}+\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 M\frac{\partial\langle u'T'\rangle}{\partial z}\right)-\frac{M}{\Lambda^2}\langle u'T'\rangle+\frac{\mu}{\rho_0}\frac{\partial^2}{\partial z^2}\langle u'T'\rangle-\frac{2\mu}{\rho_0}\frac{\langle u'T'\rangle}{\lambda^2} \tag{49}$$

$$\frac{\partial\langle w'T'\rangle}{\partial t}=-\langle w'w'\rangle\frac{\partial\bar{T}}{\partial z}+\frac{3}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 M\frac{\partial\langle w'T'\rangle}{\partial z}\right)-\frac{M}{\Lambda^2}\langle w'T'\rangle+\frac{\mu}{\rho_0}\frac{\partial^2}{\partial z^2}\langle w'T'\rangle-\frac{2}{\rho_0}\frac{\langle w'T'\rangle}{\lambda^2}-\frac{(\mu+\mu^*)}{\rho_0}\times\left\{\langle T'w'\rangle\frac{\partial}{\partial z}\left(\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial z}\right)-\frac{\langle T'w'\rangle}{\rho_0^{\,2}}\left(\frac{\partial\rho_0}{\partial z}\right)^2\right\}+\frac{g}{T_0}\langle T'^2\rangle \tag{50}$$

$$\frac{\partial\langle T'^2\rangle}{\partial t}=-2\langle w'T'\rangle\frac{\partial\bar{T}}{\partial z}+\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 M\frac{\partial}{\partial z}\langle T'^2\rangle\right)+\frac{\mu}{\rho_0}\frac{\partial^2}{\partial z^2}\langle T'^2\rangle-\frac{2\mu}{\rho_0}\frac{\langle T'^2\rangle}{\lambda^2} \tag{51}$$

Here, again, $M=\Lambda K^{1/2}$ where $K=\langle u'u'\rangle+\langle v'v'\rangle+\langle w'w'\rangle$.

About a year ago, the author and his colleagues presented some initial calculations of the generation of turbulence in a thin shear layer, some 2000 ft thick, using a simplified set of equations derived from those just given.[9] The model parameters for these calculations were $C_1=2.5$, $C_2=0.125$, and $C_3=0.064$. For other details of this calculation, the reader is referred to the original paper.

Calculations using the complete set of equations given above are now under way at Aeronautical Research Associates of Princeton. Figures 9–11 give some idea of the type of results that may be expected.

Figure 9 shows a distribution of mean velocity in an atmosphere that was at rest at time $t=0$ and forced into motion by a body force that produced, at the end of 3000 sec, the mean velocity profile shown in the figure. The initial distribution of temperature was very close to that shown by the solid line marked $\bar{T}$ in Fig. 11. It is seen that the lapse rate was stable below 20,000 ft and unstable above that altitude. The altitude of 20,000 ft corresponded to the maximum of the atmospheric forcing function $X(z,t)$ so that the atmosphere was unstable in the upper half and stable in the lower half of the shear layer.

The distribution of the shear correlation $\langle u'w'\rangle$ as computed for the same time as the mean velocity profile just discussed is also shown in Fig. 9. It will be seen that the stability of the atmosphere below 20,000 ft and the instability of the atmosphere above 20,000 ft caused an asymmetry in the turbulent shear distribution. This caused the asymmetry of the mean velocity distribution that is shown. The effect of atmospheric stability is to

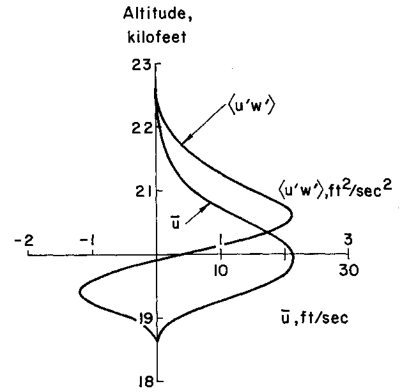

Fig. 9 u **and** $\langle u'w'\rangle$ **at 3000 sec.**

increase the turbulent transport of momentum on the unstable side of the shear layer and decrease it on the stable side of the layer from what it would be in a neutrally stable case, but the effect is not nearly as pronounced on the transport of momentum as it is on the transport of heat, as will be seen in what follows.

In Fig. 10, we show the distribution of $\langle T'^2\rangle$ and the distribution of $K=\langle u'u'\rangle+\langle v'v'\rangle+\langle w'w'\rangle$ at the time (3000 sec) for which the mean velocity and shear correlation profiles we have discussed were computed. It is interesting to note the very much larger production and spread of turbulence on the unstable side of the shear layer.

In Fig. 11, we show the mean temperature profiles as given by $\bar{T}(z)$ for two times, very early in the motion (300 sec) and at

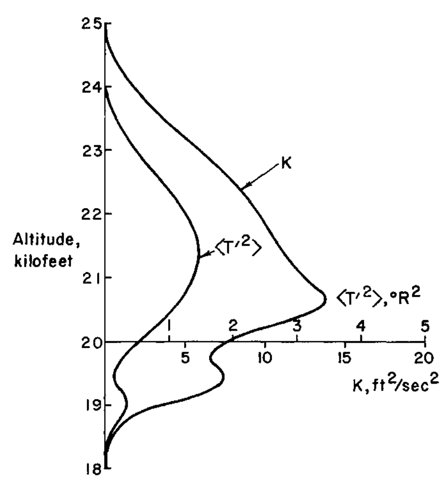

Fig. 10 $\langle T'^2\rangle$ **and** K **at 3000 sec.**

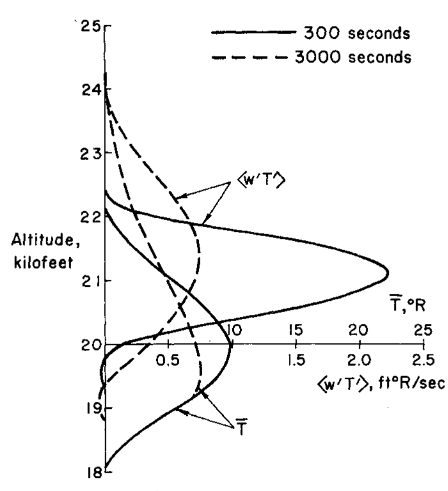

Fig. 11 $\bar{T}$ **and** $\langle w'T'\rangle$ **at 300 and 3000 sec.**

the same time (3000 sec) as that for which the other results have been given. The interesting result contained in this figure is the very different behavior of the heat-transfer correlation $\langle w'T' \rangle$ on the unstable and stable sides of the shear layer. In effect, we see that it is, for the conditions of this numerical experiment, almost impossible to create a turbulent heat-transfer correlation on the stable side of the shear layer; conversely, it is very easy to produce large heat-transfer correlations on the unstable side of the shear layer.

We can see from the results just presented that, by using the equations for the actual correlations themselves in computing the turbulent heat transfers, the effect of atmospheric stability on this quantity is automatically and properly factored into our calculations, and no alteration of the model is necessary to account for the effects of stability, as in the case with conventional mixing length formulations.

There are many more insights into the nature of atmospheric turbulence that may be gleaned by performing numerical experiments using the modeling technique discussed here. We are, at present, just getting started on the calculaton of a number of interesting atmospheric flows which we hope to be able to report on in detail in the near future.

Future Directions and Possibilities

It is hoped that the two very brief glimpses given in this paper of applications of the method of invariant modeling to shear flows will give those who are interested in such matters some idea of the advances that may be made in the art of computing turbulent flows. These advances have been made possible within the last few years by developments in the computer industry. It is the author's opinion that some form of modeling closure of the equations for the second-order turbulent correlations in strongly sheared flowfields represents, at the present time, our best hope for an advance in our ability to predict characteristics of such flows. For another method of the type discussed here, the reader is referred to the work of Harlow et al.[14,15] I feel strongly that the most fruitful approach of this kind must make use of the full set of equations in each case and that simplifications based on using a contracted form of Eq. (7), i.e., an equation for only the sum of all three components of turbulence $(\langle u'u' \rangle + \langle v'v' \rangle + \langle w'w' \rangle)$ (Refs. 16–18), although they do save some machine time, will not result in a method which is in any way as powerful or general as one similar to that which we have discussed here.

What are some of the problems that may be tackled with the aid of methods akin to the invariant modeling technique which my colleagues and I are trying to develop? Some that come immediately to mind are 1) isolated vortex decay in the presence of axial components of mean velocity; 2) turbulent heat transfer within rotating systems (the arc jet chamber is a good example of such a problem); 3) the dispersal of pollutants from line and point sources in the atmosphere (it is very easy to derive the equations for the transport of some gaseous pollutant in the atmosphere by the technique we have presented); 4) the behavior of a low Mach number jet of a foreign gas as it mixes with a given atmosphere; 5) the problem of the atmospheric boundary layer. These are but a few of the problems which the author and his colleagues have considered in some detail. We are, at present, working actively on several of them. Less far along is the application of the method to the compressible boundary layer, to the problem of compressible transition, and to the problem of aerodynamic noise.

Even further down the line might be the development of a "spectral" version of the present method. If such a method could be developed, it would eliminate the need for one to know or estimate the scale Λ. The application of such an extended method to practical problems will require, however, yet another generation of computers. It certainly would require more patience to develop than this writer has left. Perhaps it will not be necessary after all, since there is a good possibility that someone may solve the problem of closure for shear layers by resorting to techniques more rational and elegant than invariant modeling.

These last techniques are a bit in the future. This writer is confident that tools are now available to study the general characteristics of a rather large group of turbulent flow problems in a much more complete and rational fashion than was possible some years ago. This step forward has been made possible, as we have pointed out several times, primarily by advances in computer technology.

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